Int. J. Solids Structures Vol. 20, No. 1, pp. 13–30, 1984
Printed in Great Britain.

0020-7683/84 $3.00 + .00
Pergamon Press Ltd.

GENERAL HELICOIDAL SHELLS UNDERGOING LARGE, ONE-DIMENSIONAL STRAINS OR LARGE INEXTENSIONAL DEFORMATIONS†

JAMES G. SIMMONDS
Department of Applied Mathematics and Computer Science, University of Virginia, Charlottesville, VA 22901, U.S.A.

(*Received* 14 *December* 1982; *in revised form* 13 *April* 1983)

Abstract—An earlied paper showed that general helicoidal shells (which include shells of revolution and general cylindrical shells as limiting cases) admit arbitrarily large, one-dimensional strain fields. In the present paper, the associated rotation and stress function fields are found. Introduction of the Euler parameters from rigid body dynamics reduces the determination of the rotation field to a linear problem. The equilibrium and compatibility equations are shown to reduce to six coupled scalar equations involving three rotation functions, three stress functions, extensional strains, stress couples, and four constants, two measuring the gross axial displacement and twist of the shell, and two measuring the net axial force and torque. One field equation is a first integral of the compatibility equations; another, a first integral of the moment equilibrium equations. Reissner's equations for the pure bending of curved tubes and Wan's equations for the gross twisting and extension of right helicoidal shells fall out as special cases. Determination of the displacement field in large inextensional bending reduces to quadratures, generalizing Reissner's result for a slit shell of revolution.

1. INTRODUCTION

The field equations of general shell theory are formidable partial differential equations. Their solutions, even for simple geometries, loads, and constitutive laws, must usually be approximated by using large computer codes. The question, "When do these partial differential equations reduce to ordinary ones?" is, curiosity aside, important for two reasons. First, since a positive answer can be expected only for shells having simple shapes, these should be shells that are relatively easy to manufacture and therefore, one hopes, of some practical use. Second, because numerical methods for the solution of partial differential equations often differ radically from those for ordinary differential equations, solutions of nonlinear, one-dimensional shell problems can provide valuable benchmarks for all-purpose computer codes.

Shells with undeformed midsurfaces that are general helicoids admit one-dimensional strain fields [1]. Of course, only special loads, boundary conditions, and material properties produce these strains. Such a case is illustrated in Fig. 1 where an elastically isotropic helical tube with a thickness variation that is identical but arbitrary in every plane through its longitudinal axis, is under a constant internal pressure p, an axial force P, and an axial torque T.

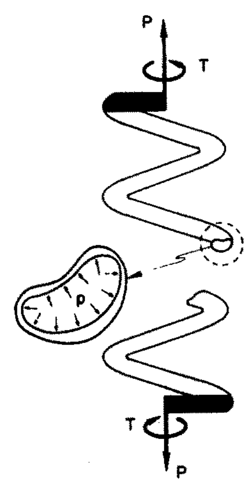

Fig. 1. A loaded helicoidal shell with a one-dimensional strain field.

†This work was supported by the National Science Foundation under grant CEE-8117103.

A helicoid has the parametric representation

$$\mathbf{r}(s,\theta)=r(s)\mathbf{e}_r(\theta)+[b\theta+z(s)]\mathbf{e}_z,\quad 0\leq s\leq L,\quad -\Theta\leq\theta\leq\Theta. \tag{1.1}$$

Here

$$\mathbf{e}_r(\theta)=\cos\theta\mathbf{e}_x+\sin\theta\mathbf{e}_y, \tag{1.2}$$

where $\{\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z\}$ is a fixed, right-handed, orthonormal Cartesian frame, r and z are functions of arc length s along a meridional cross-section (θ = constant), and b is the constant pitch of the helicoid.

In classical shell theory, the extensional and bending strains are linear combinations of the differences between, respectively, the metric and curvature coefficients of the deformed and undeformed midsurface. If, in a helicoidal shell, these strains are functions of s only, then the deformed mid-surface is the helicoid[1]

$$\bar{\mathbf{r}}(s,\theta)=\bar{r}(s)\mathbf{e}_r(\lambda\theta-\tau(s))+[\bar{b}\theta+\bar{z}(s)]\mathbf{e}_z, \tag{1.3}$$

where $\bar{r}$, $\bar{z}$, and τ are arbitrary, sufficiently smooth functions of s, and λ and $\bar{b}$ are arbitrary constants.

The path to the displacement form of the field equations is clear, but tedious and unenlightening. The three equations of moment equilibrium are used to express the two transverse shear stress resultants and the skew part of the stress resultant in terms of stress couples and strains, and stresses are expressed in terms of strains via constitutive laws. Finally, strains are expressed in terms of $\bar{r}$, $\bar{z}$, τ, λ, and $\bar{b}$ using (1.1) and (1.3) and everything is substituted into the three equations of force equilibrium. This produces three equations for $\bar{r}$, $\bar{z}$, and τ. The constants λ and $\bar{b}$ are determined when specific boundary value problems are solved.

There are several drawbacks to a displacement formulation. First, whenever nearly inextensional deformation occurs, the field equations become ill-conditioned. Second, as Reissner's studies of shells of revolution[2–4] and curved tubes[5–7] have shown, a dual rotation-stress function formulation, at least for these special cases of one-dimensional deformation, leads to simple, lower order equations. And third, if we wish to allow for transverse shear and "twisting" strains[8]. This approach leads to the field equations developed herein, which may be thought of as integrated forms of the displacement field equations. However, unlike the latter, they exhibit the static-geometric duality when linearized.

2. THE GEOMETRY OF DEFORMATION

In what follows a prime or dot denotes, respectively, the total or partial derivative with respect to s or θ. At each point of the undeformed midsurface (1.1) the base vectors

$$\mathbf{a}_s=\mathbf{r}'=\cos\phi\mathbf{e}_r+\sin\phi\mathbf{e}_z \tag{2.1}$$

$$\mathbf{a}_\theta=\dot{\mathbf{r}}=r\mathbf{e}_\theta+b\mathbf{e}_z, \tag{2.2}$$

and the unit normal

$$\mathbf{n}=a^{-1/2}(-r\sin\phi\mathbf{e}_r-b\cos\phi\mathbf{e}_\theta+r\cos\phi\mathbf{e}_z), \tag{2.3}$$

form a non-orthogonal frame $\{\mathbf{a}_i\}\equiv\{\mathbf{a}_s,\mathbf{a}_\theta,\mathbf{n}\}$, $i=1,2,3$. In (2.1) to (2.3)

$$\cos\phi=r',\quad \sin\phi=z', \tag{2.4}$$

$$\mathbf{e}_\theta=-\sin\theta\mathbf{e}_x+\cos\theta\mathbf{e}_y, \tag{2.5}$$

and

$$a\equiv|\mathbf{a}_s\times\mathbf{a}_\theta|^2=r^2+b^2\cos^2\phi. \tag{2.6}$$

The deformation that carries $\{\mathbf{a}_i\}$ at (s, θ) into the deformed frame $\{\bar{\mathbf{a}}_i\} \equiv \{\bar{\mathbf{a}}_s, \bar{\mathbf{a}}_\theta, \bar{\mathbf{n}}\}$ associated with (1.3) may be decomposed as follows. First, $\{\mathbf{a}_i\}$ is rotated rigidly into an intermediate frame $\{\mathbf{A}_i\} \equiv \{\mathbf{A}_s, \mathbf{A}_\theta, \mathbf{N}\}$, characterized by a proper orthogonal tensor (or *rotator*) $\mathbf{Q}$: i.e.

$$\mathbf{A}_i = \mathbf{Q}\cdot\mathbf{a}_i. \tag{2.7}$$

Next, the frame $\{\mathbf{A}_i\}$ is given an additional rigid body rotation, characterized by a shearing twist strain rotator $\mathbf{\Gamma}$ that brings $\mathbf{N}$ into coincidence with $\bar{\mathbf{n}}$.

Finally, a symmetric, positive definite stretch tensor $\mathbf{V}$ sends the rotated frame $\mathbf{\Gamma}\cdot\{\mathbf{A}_i\}$ into the deformed frame $\{\bar{\mathbf{a}}_i\}$. In terms of the unsymmetric strain tensor

$$\mathbf{E} \equiv \mathbf{V}\cdot\mathbf{\Gamma}, \tag{2.8}$$

we have

$$\bar{\mathbf{a}}_i = \mathbf{E}\cdot\mathbf{Q}\cdot\mathbf{a}_i = \mathbf{E}\cdot\mathbf{A}_i. \tag{2.9}$$

The rotator that sends $\{\mathbf{a}_i\}$ into $\{\mathbf{A}_i\}$ has the representation[9, 13]

$$\mathbf{Q} = (2\mu^2 - 1)\mathbf{I} + 2\boldsymbol{\beta}\boldsymbol{\beta} + 2\mu\boldsymbol{\beta}\times. \tag{2.10}$$

Here,

$$\mu = \cos(\beta/2) \tag{2.11}$$

and

$$\boldsymbol{\beta} = \sin(\beta/2)\mathbf{e}, \tag{2.12}$$

where $\mathbf{e}$ is a unit vector along the axis of rotation and β is the angle of rotation, reckoned positive by the right-hand rule; $\boldsymbol{\beta}\boldsymbol{\beta}$ denotes a direct product, μ and (the Cartesian components of) $\boldsymbol{\beta}$ are called Euler parameters.† Use of $\boldsymbol{\beta}$ and μ in place of the finite rotation vector $\boldsymbol{\psi} = 2\boldsymbol{\beta}/\mu$, which is used in[8] and[10], leads to *linear* equations for the determination of the θ-dependence of μ and $\boldsymbol{\beta}$. Analogous to $\mathbf{Q}$, $\mathbf{\Gamma}$ may be represented in terms of a finite shear-twist vector $\boldsymbol{\gamma}$. (See [8] for more details.)

3. THE ROTATION FIELD

The bending of the shell may be described by two vectors[8, 11, 12] $\mathbf{k}_s$ and $\mathbf{K}_\theta$, defined, respectively, as the axial vectors of the skew tensors $\mathbf{Q}'\cdot\mathbf{Q}^T$ and $\mathbf{Q}^{\cdot}\cdot\mathbf{Q}^T$. In what follows it proves to be convenient to work with the rotated frame $\{\mathbf{E}_i\} \equiv \{\mathbf{E}_r, \mathbf{E}_\theta, \mathbf{E}_z\}$, where

$$\mathbf{E}_i = \mathbf{Q}\cdot\mathbf{e}_i, \{\mathbf{e}_i\} \equiv \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}. \tag{3.1}$$

Note that

$$[\mathbf{A}_i\cdot\mathbf{E}_j] = [\mathbf{a}_i\cdot\mathbf{e}_j] = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & r & b \\ -a^{-1/2}r\sin\phi & -a^{-1/2}b\cos\phi & a^{-1/2}r\cos\phi \end{bmatrix} \tag{3.2}$$

$$\frac{1}{a}[\mathbf{A}^i\cdot\mathbf{E}_j] = [\mathbf{a}^i\cdot\mathbf{e}_j] = \begin{bmatrix} (b^2+r^2)\cos\phi & -rb\sin\phi & r^2\sin\phi \\ -b\sin\phi\cos\phi & r & b\cos^2\phi \\ -a^{1/2}r\sin\phi & -a^{1/2}b\cos\phi & r\cos\phi \end{bmatrix}. \tag{3.3}$$

†I thank Prof. Harold Morton for pointing out to me the advantages of the Euler parameters.

where $\mathbf{A}^i \cdot \mathbf{A}_j = \mathbf{a}^i \cdot \mathbf{a}_j = \delta_j^i$, the Kronecker delta. As

$$\mathbf{e}_i{}' = \mathbf{0} \text{ and } \mathbf{e}_i^{\cdot} = \mathbf{e}_z \times \mathbf{e}_i, \tag{3.4}$$

it follows that

$$\mathbf{E}_i{}' = \mathbf{Q}' \cdot \mathbf{e}_i = \mathbf{Q}' \cdot \mathbf{Q}^T \cdot \mathbf{E}_i = \mathbf{K}_s \times \mathbf{E}_i \tag{3.5}$$

$$\mathbf{E}_i^{\cdot} = \mathbf{Q}^{\cdot} \cdot \mathbf{e}_i + \mathbf{Q} \cdot \mathbf{e}_i^{\cdot} = (\mathbf{K}_\theta + \mathbf{E}_z) \times \mathbf{E}_i. \tag{3.6}$$

Equation (3.70) of [8], when expressed in terms of $\boldsymbol{\beta}$ and μ, takes the form

$$\mathbf{K}_s = 2(\mu \boldsymbol{\beta}' - \mu' \boldsymbol{\beta} \times \boldsymbol{\beta}') \tag{3.7}$$

$$\mathbf{K}_\theta = 2(\mu \boldsymbol{\beta}^{\cdot} - \mu^{\cdot} \boldsymbol{\beta} + \boldsymbol{\beta} \times \boldsymbol{\beta}^{\cdot}). \tag{3.8}$$

Let

$$\boldsymbol{\beta} = \beta_i \mathbf{E}_i = \beta_i \mathbf{e}_i \tag{3.9}$$

and, noting the tensor form $\mathbf{K}_\alpha = K_{\alpha\beta}\mathbf{A}^\gamma \times \mathbf{N} + K_\alpha \mathbf{N} = \epsilon_{\gamma\beta} K_\alpha{}^{\beta} \mathbf{A}^\gamma + K_\alpha \mathbf{N}$, set

$$\begin{aligned} \mathbf{K}_s &= a^{1/2}(K_{s.}{}^{\theta}\mathbf{A}^s - K_{s.}{}^{s}\mathbf{A}^\theta) + K_s \mathbf{N} \\ &= \tilde{K}_{s\theta}\mathbf{E}_r - \tilde{K}_{ss}\mathbf{E}_\theta + \tilde{K}_s \mathbf{E}_z \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mathbf{K}_\theta &= a^{1/2}(K_{\theta.}{}^{\theta}\mathbf{A}^s - K_{\theta.}{}^{s}\mathbf{A}^\theta) + K_\theta \mathbf{N} \\ &= \tilde{K}_{\theta\theta}\mathbf{E}_r - \tilde{K}_{\theta s}\mathbf{E}_\theta + \tilde{K}_\theta \mathbf{E}_z. \end{aligned} \tag{3.11}$$

Our task is to determine from (3.8) the explicit θ-dependence of μ and $\boldsymbol{\beta}$ given that $(K_{s.}{}^{s}, K_{s.}{}^{\theta}, \ldots)$ [and hence $(\tilde{K}_{ss}, \tilde{K}_{s\theta}, \ldots)$] are functions of s only.

First, we take the dot product of both sides of (3.8) with $\boldsymbol{\beta}$. Noting that

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta} + \mu^2 = 1 \tag{3.12}$$

and hence that

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta}^{\cdot} + \mu\mu^{\cdot} = 0, \tag{3.13}$$

we obtain

$$\mu^{\cdot} = -\frac{1}{2}\mathbf{K}_\theta \cdot \boldsymbol{\beta}. \tag{3.14}$$

Next, we use (3.12), (3.14), and the fact that the equation

$$\mathbf{x} + \mathbf{a} \times \mathbf{x} = \mathbf{b} \tag{3.15}$$

has the unique solution ([13], Exercise 1.9)

$$\mathbf{x} = \frac{\mathbf{b} + (\mathbf{a} \cdot \mathbf{b})\mathbf{a} + \mathbf{b} \times \mathbf{a}}{1 + \mathbf{a} \cdot \mathbf{a}} \tag{3.16}$$

to obtain from (3.8),

$$\boldsymbol{\beta}^{\cdot} = \frac{1}{2}(\mathbf{K}_\theta \mu + \mathbf{K}_\theta \times \boldsymbol{\beta}). \tag{3.17}$$

If we introduce the notation

$$\mathcal{D}_\theta\boldsymbol{\beta} \equiv \beta_i^{\cdot}\mathbf{E}_i, \tag{3.18}$$

then by (3.6)

$$\boldsymbol{\beta}^{\cdot} = \mathcal{D}_\theta\boldsymbol{\beta} + (\mathbf{K}_\theta + \mathbf{E}_z)\times\boldsymbol{\beta}, \tag{3.19}$$

and (3.17) takes the final form

$$\mathcal{D}_\theta\boldsymbol{\beta} = \frac{1}{2}\mathbf{K}_\theta\mu - \left(\frac{1}{2}\mathbf{K}_\theta + \mathbf{E}_z\right)\times\boldsymbol{\beta}. \tag{3.20}$$

Equations (3.14) and (3.20) together form a linear, fourth-order skew system that we may write, schematically, as

$$\mathcal{D}_\theta\begin{bmatrix}\boldsymbol{\beta}\\ \mu\end{bmatrix} = \begin{bmatrix}-\mathbf{h}\times & \mathbf{k}\\ -\mathbf{k}\cdot & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{\beta}\\ \mu\end{bmatrix},\quad \mathbf{k} \equiv \frac{1}{2}\mathbf{K}_\theta,\quad \mathbf{h} \equiv \mathbf{k} + \mathbf{E}_z. \tag{3.21}$$

As the components of $\mathbf{h}$ and $\mathbf{k}$ are functions of s only, (3.21) is, formally, a constant coefficient equation. Being skew it has solutions of the form

$$\beta_j = e^{ip(s)\theta}b_j(s),\quad \mu = e^{ip(s)\theta}m(s), \tag{3.22}$$

where

$$\begin{bmatrix}ip\mathbf{1} + \mathbf{h}\times & \mathbf{k}\\ \mathbf{k}\cdot & ip\end{bmatrix}\begin{bmatrix}\mathbf{b}\\ m\end{bmatrix} = 0,\quad \mathbf{b} = b_j\mathbf{E}_j, \tag{3.23}$$

and p satisfies the characteristic polynomial

$$p^4 - (\mathbf{h}\cdot\mathbf{h} + \mathbf{k}\cdot\mathbf{k})p^2 + (\mathbf{h}\cdot\mathbf{k})^2 = 0. \tag{3.24}$$

Let

$$q = \frac{1}{2}(1 - |\mathbf{h}+\mathbf{k}|) = \frac{1}{2}(1 - |\mathbf{K}_\theta + \mathbf{E}_z|). \tag{3.25}$$

Then the four roots of (3.24) are $\pm q$ and $\pm(1-q)$. Note that $q \to 0$ as $\mathbf{K}_\theta \to \mathbf{0}$.

Proceeding as we did from (3.8) to (3.20), we find that (3.5), (3.7), and (3.12) imply

$$\mu' = -\frac{1}{2}\mathbf{K}_s\cdot\boldsymbol{\beta} \tag{3.26}$$

$$\mathcal{D}_s\boldsymbol{\beta} \equiv \beta_j'\mathbf{E}_j = \frac{1}{2}(\mu\mathbf{K}_s - \mathbf{K}_s\times\boldsymbol{\beta}). \tag{3.27}$$

Introduction of (3.10) and (3.22) shows that p is a constant. Hence, by (3.25), $\mathbf{K}_\theta$ must be of the form

$$\mathbf{K}_\theta = -\mathbf{E}_z + (1-2q)\mathbf{U},\quad |\mathbf{U}| = 1, \tag{3.28}$$

where the components of $\mathbf{U}$ in the frame $\{\mathbf{E}_i\}$ are functions of s only.

Henceforth, with little loss in generality and considerable saving of algebra, we shall discard the solutions associated with the roots $\pm(1-q)$. The reason is this.

Let $\boldsymbol{\beta}_B$ and μ_B be any vector and scalar functions independent of θ, and let $(\hat{\boldsymbol{\beta}}, \hat{\mu})$ be *any* non-trivial solution of (3.21). Then, as may be verified by direct substitution,

$$\boldsymbol{\beta} = \hat{\mu}\boldsymbol{\beta}_B + \mu_B\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_B \times \hat{\boldsymbol{\beta}}, \quad \mu = \mu_B\hat{\mu} - \boldsymbol{\beta}_B \cdot \hat{\boldsymbol{\beta}} \tag{3.29}$$

also satisfies (3.21). (This is the *nonlinear* addition formula for successive finite rotations given, for example, in [9].) Indeed, *every* solution of (3.21) is of the form (3.29), for we may always choose $\boldsymbol{\beta}_B$ and μ_B so that $\boldsymbol{\beta}$ and μ meet any initial conditions. Here we are appealing to the existence and uniqueness theorem for a system of linear ordinary differential equations with coefficients that depend analytically (as we shall assume) on a parameter (s in our case): *two solutions that agree at a point exist and agree everywhere.* But (3.29) must also satisfy (3.26) and (3.27). Substituting, we find that, in fact, $\boldsymbol{\beta}_B$ and μ_B must be constant.

In summary, all solutions to (3.2) are generated by superimposing, according to (3.29), a rigid body motion on any non-trivial solution of (3.21). We shall take that non-trivial solution to be a certain linear combination of the solutions associated with $\pm q$, adjusting any arbitrary functions of s so that $\boldsymbol{\beta} \to \mathbf{0}$ as $\mathbf{K}_s, \mathbf{K}_\theta \to \mathbf{0}$. Should rigid body solutions be needed, they can be gotten from (3.29).

To complete the solution of the eigenvalue problem (3.23), we take μ, which is real, in the form

$$\mu = \Re C(s)e^{iq\theta} = A(s)\cos[q\theta + \sigma(s)], \tag{3.30}$$

where $C(s) = A(s)e^{i\sigma(s)}$ and $\Re$ denotes "the real part of". From (3.21), (3.23), and (3.28),

$$ip\beta + \left[\frac{1}{2}\mathbf{E}_z + \left(\frac{1}{2} - q\right)\mathbf{U}\right] \times \boldsymbol{\beta} = m\left[\frac{1}{2}\mathbf{E}_z - \left(\frac{1}{2} - q\right)\mathbf{U}\right]. \tag{3.31}$$

Writing the solution for $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} = \Re \mathbf{b}e^{iq\theta}, \tag{3.32}$$

we have, by (3.15), (3.16), and (3.30),

$$\mathbf{b} = \frac{C[\mathbf{U} \times \mathbf{E}_z + i(\mathbf{U} + \mathbf{E}_z)]}{1 + \mathbf{U}\cdot\mathbf{E}_z}. \tag{3.33}$$

As $\boldsymbol{\beta}$ has the same components in the frames $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$, symmetry dictates that $\mathbf{U} = \mathbf{e}_z$.

To further reduce (3.32) set

$$\mathbf{U} = \mathbf{e}_z = \sin\psi(\cos\alpha\mathbf{E}_r + \sin\alpha\mathbf{E}\theta) + \cos\psi\mathbf{E}_z \tag{3.34}$$

$$A = B\cos(\psi/2), \tag{3.35}$$

where ψ, α, and B are functions of s. Then

$$\mathbf{e}_z \times \mathbf{E}_z = \sin\psi(\sin\alpha\mathbf{E}_r - \cos\alpha\mathbf{E}_\theta) \equiv -2\sin(\psi/2)\cos(\psi/2)\bar{\mathbf{e}}_\theta \tag{3.36}$$

$$\begin{aligned}\mathbf{e}_z + \mathbf{E}_z &= 2\cos(\psi/2)[\sin(\psi/2)(\cos\alpha\mathbf{E}_r + \sin\alpha\mathbf{E}_\theta) + \cos(\psi/2)\mathbf{E}_z] \\ &\equiv 2\cos(\psi/2)\boldsymbol{\mu}\end{aligned} \tag{3.37}$$

and

$$\boldsymbol{\beta} = -[\sin(\psi/2)\cos(q\theta + \sigma)\bar{\mathbf{e}}_\theta + \sin(q\theta + \sigma)\boldsymbol{\mu}]. \tag{3.38}$$

The condition $\boldsymbol{\beta}\cdot\boldsymbol{\beta} + \mu^2 = 1$ yields $B = 1$. Thus

$$\mu = \cos(\psi/2)\cos q\theta \tag{3.39}$$

and in the frame $(\{\mathbf{E}_i\}$,

$$\boldsymbol{\beta} = -\sin(\psi/2)[\sin(q\theta+\sigma-\alpha)\mathbf{E}_r + \cos(q\theta+\sigma-\alpha)\mathbf{E}_\theta] - \cos(\psi/2)\sin(q\theta+\sigma)\mathbf{E}_z. \quad (3.40)$$

If $\mathbf{K}_s, \mathbf{K}_\theta \to \mathbf{0}$, ψ, σ, $q \to 0$ and $\{\mathbf{E}_i\} \to \{\mathbf{e}_i\}$, but α remains indeterminant.

We infer from (3.40) that $\mathbf{Q} = \mathbf{R}\cdot\mathbf{S}$, where $\mathbf{R}$ and $\mathbf{S}$ are rotators with associated Euler parameters

$$\boldsymbol{\beta}_R = \sin(\psi/2)[\sin(\alpha/2)\mathbf{e}_r - \cos(\alpha/2)\mathbf{e}_\theta] - \cos(\psi/2)\sin(\alpha/2)\mathbf{e}_z \quad (3.41)$$

$$\mu_R = \cos(\alpha/2)\cos(\psi/2) \quad (3.42)$$

$$\boldsymbol{\beta}_S = -\sin(q\theta+\gamma/2)\mathbf{e}_z, \quad \mu_S = \cos(q\theta+\gamma/2), \quad (3.43, 3.44)$$

where

$$\gamma = 2\sigma - \alpha \quad (3.45)$$

and

$$\{\bar{\mathbf{e}}_r, \bar{\mathbf{e}}_\theta, \mathbf{e}_z\} \equiv \{\bar{\mathbf{e}}_i\} = \mathbf{S}\cdot\mathbf{e}_i. \quad (3.46)$$

As $\mathbf{S}$ is merely a rotation about $-\mathbf{e}_z$ through an angle $2q\theta + \gamma$,

$$\bar{\mathbf{e}}_r = \cos(2q\theta+\gamma)\mathbf{e}_r - \sin(2q\theta+\gamma)\mathbf{e}_\theta \quad (3.47)$$

$$\bar{\mathbf{e}}_\theta = \sin(2q\theta+\gamma)\bar{\mathbf{e}}_r + \cos(2q\theta+\gamma)\mathbf{e}_\theta, \quad (3.48)$$

from which follow

$$\bar{\mathbf{e}}_i' = -\gamma'\mathbf{e}_z \times \bar{\mathbf{e}}_i, \quad \dot{\bar{\mathbf{e}}}_i = (1-2q)\mathbf{e}_z \times \bar{\mathbf{e}}_i. \quad (3.49, 3.50)$$

The relation $\mathbf{E}_i = \mathbf{R}\cdot\bar{\mathbf{e}}_i$ takes the explicit form

$$\mathbf{E}_r = \cos\alpha\cos\psi\bar{\mathbf{e}}_r - \sin\alpha\bar{\mathbf{e}}_\theta + \cos\alpha\sin\psi\mathbf{e}_z \quad (3.51)$$

$$\mathbf{E}_\theta = \sin\alpha\cos\psi\bar{\mathbf{e}}_r + \cos\alpha\bar{\mathbf{e}}_\theta + \sin\alpha\sin\psi\mathbf{e}_z \quad (3.52)$$

$$\mathbf{E}_z = -\sin\psi\bar{\mathbf{e}}_r + \cos\psi\mathbf{e}_z. \quad (3.53)$$

Note that (3.40) for $\boldsymbol{\beta}$ is recovered from (3.41) to (3.44) by applying (3.29) with $(\boldsymbol{\beta}_R, \mu_R)$ and $(\boldsymbol{\beta}_S, \mu_S)$ in place of $(\hat{\boldsymbol{\beta}}, \hat{\mu})$ and $(\boldsymbol{\beta}_B, \mu_B)$, respectively.

The decomposition $\mathbf{Q} = \mathbf{R}\cdot\mathbf{S}$ is useful because μ_R and the components of $\boldsymbol{\beta}_R$ in the basis $\{\bar{\mathbf{e}}_i\}$ or $\{\mathbf{E}_i\}$ are functions of s only. At the same time, the basis $\{\bar{\mathbf{e}}_i\}$ is almost as simple as the basis $\{\mathbf{e}_i\} = \{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$.

4. THE BENDING FIELDS

It follows immediately from (3.28) and (3.34) that

$$\mathbf{K}_\theta = (1-2q)\mathbf{e}_z - \mathbf{E}_z. \quad (4.1)$$

Using (3.12), (3.15), and (3.16), we solve (3.27) for $\mathbf{K}_s$ to get

$$\frac{1}{2}\mathbf{K}_s = \mu\mathcal{D}_s\boldsymbol{\beta} + \mu^{-1}(\boldsymbol{\beta}\cdot\mathcal{D}_s\boldsymbol{\beta})\boldsymbol{\beta} + \mathcal{D}_s\boldsymbol{\beta}\times\boldsymbol{\beta}. \quad (4.2)$$

From (3.27), (3.40), and (3.51)–(3.53) follows

$$\mathbf{K}_s = -(\gamma'\mathbf{e}_z + \alpha'\mathbf{E}_z + \gamma'\bar{\mathbf{e}}_\theta). \tag{4.3}$$

The simplicity of (4.1) and (4.3) seems remarkable.

To get the components of $\mathbf{K}_s$ and $\mathbf{K}_\theta$ in the frame $\{\mathbf{A}^i\}$, let $\{\mathbf{s}_i\} = \{\mathbf{e}_z, \mathbf{E}_z, \bar{\mathbf{e}}_\theta\}$. Then from (3.2), (3.34) and (3.36)

$$[\mathbf{A}_i \cdot \mathbf{s}_j] = \begin{bmatrix} \sin\Phi & \sin\phi & -\sin\alpha\cos\phi \\ r\sin\psi\sin\alpha + b\cos\psi & b & r\cos\alpha \\ \cos\Xi & a^{-1/2}r\cos\phi & \cos Z \end{bmatrix}, \tag{4.4}$$

where

$$\sin\Phi = \mathbf{A}_s \cdot \mathbf{e}_z = \cos\phi\sin\psi\cos\alpha + \sin\phi\cos\psi \tag{4.5}$$

$$\cos\Xi = \mathbf{N}\cdot\mathbf{e}_z = a^{-1/2}[r(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\alpha) - b\cos\phi\sin\psi\sin\alpha] \tag{4.6}$$

$$\cos Z = \mathbf{N}\cdot\bar{\mathbf{e}}_\theta = a^{-1/2}(r\sin\phi\sin\alpha - b\cos\phi\cos\alpha). \tag{4.7}$$

It now follows from (3.2), (3.10), (3.11), (4.1), (4.3), and (4.4) that

$$\mathbf{A}_s \cdot \mathbf{K}_s = a^{1/2}K_{s.}{}^{\theta} = -\gamma'\sin\Phi - \alpha'\sin\phi + \psi'\sin\alpha\cos\phi \tag{4.8}$$

$$-\mathbf{A}_\theta \cdot \mathbf{K}_s = a^{1/2}K_{s.}{}^{s} = \gamma'(r\sin\psi\sin\alpha + b\cos\psi) + \alpha' b + \psi' r\cos\alpha \tag{4.9}$$

$$\mathbf{A}_s \cdot \mathbf{K}_\theta = a^{1/2}K_{\theta.}{}^{\theta} = (1-2q)\sin\Phi - \sin\phi \tag{4.10}$$

$$-\mathbf{A}_\theta \cdot \mathbf{K}_\theta = a^{1/2}K_{\theta.}{}^{s} = b - (1-2q)(r\sin\psi\sin\alpha + b\cos\psi) \tag{4.11}$$

$$\mathbf{N}\cdot\mathbf{K}_s = K_s = -\gamma'\cos\Xi - a^{-1/2}\alpha' r\cos\phi - \psi'\cos Z \tag{4.12}$$

$$\mathbf{N}\cdot\mathbf{K}_\theta = K_\theta = (1-2q)\cos\Xi - a^{-1/2}r\cos\phi. \tag{4.13}$$

5. COMPATIBILITY

We introduce extensional-shear strain vectors by setting

$$\bar{\mathbf{a}}_s = \bar{\mathbf{r}}' = \mathbf{A}_s + \mathbf{\Gamma}_s, \bar{\mathbf{a}}_\theta = \bar{\mathbf{r}}^{\cdot} = \mathbf{A}_\theta + \mathbf{\Gamma}_\theta. \tag{5.1}$$

Then $(\bar{\mathbf{r}}^{\cdot})' = (\bar{\mathbf{r}}')^{\cdot}$ implies that

$$(\mathbf{A}_\theta + \mathbf{\Gamma}_\theta)' = (\mathbf{A}_s + \mathbf{\Gamma}_s)^{\cdot}. \tag{5.2}$$

A first integral of (5.2) follows immediately from (1.3):

$$(\mathbf{a}_\theta + \mathbf{\Gamma}_\theta)\cdot\mathbf{e}_z = \bar{b}. \tag{5.3}$$

Two more scalar compatibility equations may be obtained by setting, in (5.2),

$$\mathbf{A}_s + \mathbf{\Gamma}_s = \bar{a}_{sr}\bar{\mathbf{e}}_r + \bar{a}_{s\theta}\bar{\mathbf{e}}_\theta + \bar{z}'\mathbf{e}_z \tag{5.4}$$

$$\mathbf{A}_\theta + \mathbf{\Gamma}_\theta = \bar{a}_{\theta r}\bar{\mathbf{e}}_r + \bar{a}_{\theta\theta}\bar{\mathbf{e}}_\theta + \bar{b}\mathbf{e}_z. \tag{5.5}$$

As all components in (5.4) and (5.5) are functions of s only, it follows from (3.49) and (3.50) that the $\bar{\mathbf{e}}_r$ and $\bar{\mathbf{e}}_\theta$ components of the resulting equations satisfy

$$\bar{a}'_{\theta r} + \gamma'\bar{a}_{\theta\theta} + (1-2q)\bar{a}_{s\theta} = 0 \tag{5.6}$$

$$\bar{a}'_{\theta\theta} - \gamma'\bar{a}_{\theta r} - (1-2q)\bar{a}_{sr} = 0. \tag{5.7}$$

We introduce mixed components of strain by setting

$$\Gamma_s = E_{s.}{}^{s}\mathbf{A}_s + E_{s.}{}^{\theta}\mathbf{A}_\theta + E_s\mathbf{N} \tag{5.8}$$

$$\Gamma_\theta = E_{\theta.}{}^{s}\mathbf{A}_s + E_{\theta.}{}^{\theta}\mathbf{A}_\theta + E_\theta\mathbf{N}. \tag{5.9}$$

and note from (3.2) and (3.51) to (3.53) that

$$\mathbf{A}_s \cdot \bar{\mathbf{e}}_r = \cos\phi\cos\psi\cos\alpha - \sin\phi\sin\psi \equiv \cos\Lambda \tag{5.10}$$

$$\mathbf{A}_\theta \cdot \bar{\mathbf{e}}_r = r\cos\psi\sin\alpha - b\sin\psi \tag{5.11}$$

$$\mathbf{N}\cdot\bar{\mathbf{e}}_r = -a^{-1/2}[r(\sin\phi\cos\psi\cos\alpha + \cos\phi\sin\psi) + b\cos\phi\cos\psi\sin\alpha] \equiv \cos\Omega. \tag{5.12}$$

With these relations and (4.4), we find that

$$\bar{a}_{sr} = (1 + E_{s.}{}^{s})\cos\Lambda + E_{s.}{}^{\theta}(r\cos\psi\sin\alpha - b\sin\psi) + E_s\cos\Omega \tag{5.13}$$

$$\bar{a}_{\theta\theta} = -E_{\theta.}{}^{s}\cos\phi\sin\alpha + (1 + E_{\theta.}{}^{\theta})r\cos\alpha + E_\theta\cos Z \tag{5.14}$$

$$\bar{a}_{s\theta} = -(1 + E_{s.}{}^{s})\cos\phi\sin\alpha + E_{s.}{}^{\theta}r\cos\alpha + E_s\cos Z \tag{5.15}$$

$$\bar{a}_{\theta r} = E_{\theta.}{}^{s}\cos\Lambda + (1 + E_{\theta.}{}^{\theta})(r\cos\psi\sin\alpha - b\sin\psi) + E_\theta\cos\Omega. \tag{5.16}$$

and (5.3) takes the more explicit form

$$E_{\theta.}{}^{s}\sin\Phi + (1 + E_{\theta.}{}^{\theta})(r\sin\psi\sin\alpha + b\cos\psi) + E_\theta\cos\Xi = \bar{b}. \tag{5.17}$$

Once angles and strains are known, $\bar{z}$, to within a rigid axial displacement, follows from (5.4), (5.8), and (4.4) as

$$\bar{z} = \int [1(+ E_{s.}{}^{s})\sin\Phi + E_{s.}{}^{\theta}(r\sin\psi\sin\alpha + b\cos\psi) + E_s\cos\Xi]\,ds. \tag{5.18}$$

To express $\bar{\mathbf{r}}$ in terms of angles and strains, we use (3.50), (5.1) and (5.2) to write

$$\bar{\mathbf{r}}^{\cdot} = (1 - 2q)^{-1}(\bar{a}_{\theta\theta}\bar{\mathbf{e}}_r^{\cdot} - \bar{a}_{\theta r}\bar{\mathbf{e}}_\theta^{\cdot}). \tag{5.19}$$

But as $\bar{a}_{\theta\theta}$ and $\bar{a}_{\theta r}$ are functions of s only, it follows from (5.14) and the compatibility conditions (5.6) and (5.7) that

$$\bar{\mathbf{r}} = (1 - 2q)^{-1}(\bar{a}_{\theta\theta}\bar{\mathbf{e}}_r - \bar{a}_{\theta r}\bar{\mathbf{e}}_\theta) + (\bar{b}\theta + \bar{z})\mathbf{e}_z, \tag{5.20}$$

where $\bar{z}$ is given by (5.18). Comparing (1.3) with (5.20) we find, with the aid of (3.47) and (3.48), that

$$\lambda = (1 - 2q) \tag{5.21}$$

$$(1 - 2q)\bar{r} = (\bar{a}_{\theta\theta}^2 + \bar{a}_{\theta r}^2)^{1/2} \tag{5.22}$$

$$\tau = \gamma + \tan^{-1}(\bar{a}_{\theta\theta}/\bar{a}_{\theta r}). \tag{5.23}$$

6. SATISFACTION OF THE FORCE EQUILIBRIUM EQUATIONS

Let $(a^{1/2}\mathbf{N}^s)^{\cdot}$ and $(a^{1/2}\mathbf{N}^\theta)'$ denote, respectively, the rate of change of force/initial length along the deformed coordinate curves s = constant and θ = constant, and let $\mathbf{p}$ denote the force/initial area acting on the shell in its deformed shape. Then the vector differential equation of force equilibrium is

$$(a^{1/2}\mathbf{N}^s)' + (a^{1/2}\mathbf{N}^\theta)^{\cdot} + a^{1/2}\mathbf{p} = \mathbf{0}. \tag{6.1}$$

Clearly, $\mathbf{e}_z$ is a distinguished direction before and after deformation. This suggests the decomposition

$$\begin{aligned}\mathbf{p} &= (\mathbf{p}\cdot\mathbf{e}_z)\mathbf{e}_z + \mathbf{e}_z \times (\mathbf{p}\times\mathbf{e}_z)\\ &\equiv (1-2q)(q_z\mathbf{e}_z + \mathbf{e}_z \times \mathbf{q})\end{aligned} \tag{6.2}$$

$$\mathbf{q} = q_\theta\bar{\mathbf{e}}_r - q_r\bar{\mathbf{e}}_\theta. \tag{6.3}$$

To obtain stress components that are functions of s only, we assume that q_z, q_r, and q_θ are functions of s only. Using (3.50), we may express (6.1) in the divergence-free form

$$\left[a^{1/2}\mathbf{N}^s + (1-2q)\left(\int_0^s q_z\,dt\right)\mathbf{e}_z\right]' + [a^{1/2}(\mathbf{N}^\theta + \mathbf{q})]^{\cdot} = \mathbf{0}. \tag{6.4}$$

This equation may be satisfied identically with a vector stress-function $\mathbf{F}$ by setting

$$a^{1/2}\mathbf{N}^s = \mathbf{F}^{\cdot} - (1-2q)\left(\int_0^s a^{1/2}q_z\,dt\right)\mathbf{e}_z \tag{6.5}$$

$$a^{1/2}\mathbf{N}^\theta = -\mathbf{F}' - a^{1/2}\mathbf{q}. \tag{6.6}$$

Let us introduce the component representation

$$\begin{aligned}\mathbf{N}^s &= N^s{}_s\mathbf{A}^s + N^s{}_\theta\mathbf{A}^\theta + Q^s\mathbf{N}\\ &\equiv \bar{N}_{sr}\bar{\mathbf{e}}_r + \bar{N}_{s\theta}\bar{\mathbf{e}}_\theta + \bar{Q}_s\mathbf{e}_z\end{aligned} \tag{6.7}$$

$$\begin{aligned}\mathbf{N}^\theta &= N^\theta{}_s\mathbf{A}^s + N^\theta{}_\theta\mathbf{A} + Q^\theta\mathbf{N}\\ &\equiv \bar{N}_{\theta r}\bar{\mathbf{e}}_r + \bar{N}_{\theta\theta}\bar{\mathbf{e}}_\theta + \bar{Q}_\theta\mathbf{e}_z.\end{aligned} \tag{6.8}$$

If the above components are to be functions of s only, then, except for null stress terms, $\mathbf{F}$ must be of the form

$$\mathbf{F} = F_r(s)\bar{\mathbf{e}}_r + F_\theta(s)\bar{\mathbf{e}}_\theta + [F_z(s) + (1-2q)C\theta]\mathbf{e}_z. \tag{6.9}$$

It follows from (1.2), (2.5), (3.47), (3.48), (6.4), and (6.8) that $2\pi C$ is the axial force per turn of the *deformed* shell along the deformed coordinate curve s = constant. From (3.49), (3.50), (6.5) and (6.6),

$$a^{1/2}\mathbf{N}^s = (1-2q)\left[-F_\theta\bar{\mathbf{e}}_r + F_r\bar{\mathbf{e}}_\theta + \left(C - \int_0^s a^{1/2}q_z\,dt\right)\mathbf{e}_z\right] \tag{6.10}$$

$$a^{1/2}\mathbf{N}^\theta = -(F_r' + \gamma' F_\theta + a^{1/2}q_\theta)\bar{\mathbf{e}}_r - (F_\theta' - \gamma' F_r - a^{1/2}q_r)\bar{\mathbf{e}}_\theta - F_z'\mathbf{e}_z. \tag{6.11}$$

7. MOMENT EQUILIBRIUM

Consider a (skewed) panel of the shell whose undeformed midsurface occupies the region $0 \le s_1 < s < s_2 \le L$, $-\Theta \le \theta_1 < \theta < \theta_2 \le \Theta$. If there are no distributed surface couples, then moment equilibrium requires that

$$\int_{s_1}^{s_2} a^{1/2}[\mathbf{M}^\theta + \bar{\mathbf{r}}\times\mathbf{N}^\theta]_{\theta_1}^{\theta_2}\,ds + \int_{\theta_1}^{\theta_2} a^{1/2}[\mathbf{M}^s + \bar{\mathbf{r}}\times\mathbf{N}^s]_{s_1}^{s_2} + \int_{\theta_1}^{\theta_2}\int_{s_1}^{s_2}(\bar{\mathbf{r}}\times\mathbf{p})a^{1/2}\,d\theta\,ds = \mathbf{0}. \tag{7.1}$$

As the size of the undeformed panel is arbitrary, this integral condition plus smoothness and (6.1) implies the differential equation

$$(a^{1/2}\mathbf{M}^s)' + (a^{1/2}\mathbf{M}^\theta)^{\cdot} + \bar{\mathbf{r}}' \times a^{1/2}\mathbf{N}^s + \bar{\mathbf{r}}^{\cdot} \times a^{1/2}\mathbf{N}^\theta = 0. \tag{7.2}$$

If we dot both sides of (7.1) with $\mathbf{e}_z$ and note that $(\mathbf{M}^\theta + \bar{\mathbf{r}} \times \mathbf{N}^s) \cdot e_z$ is independent of θ because the strains are stresses are, then we may conclude that, in the limit as $\theta_2 \to \theta_1$,

$$\left[a^{1/2}(\mathbf{M}^s + \bar{\mathbf{r}} \times \mathbf{N}^s) + \int_0^s (\bar{\mathbf{r}} \times \mathbf{p}) a^{1/2}\, dt\right] \cdot \mathbf{e}_z = (1-2q)B, \tag{7.3}$$

where B, a constant, is the net axial moment per turn of the *deformed* helicoidal shell acting over any deformed coordinate curve s = constant. This equation is a first integral of (7.2). Two more scalar equations of moment equilibrium may be obtained by introducing components of $\mathbf{M}^s$ and $\mathbf{M}^\theta$. Noting the tensor form $\mathbf{M}^\alpha = M^{\alpha\beta}\mathbf{A}_\beta \times \mathbf{N} + M^\alpha \mathbf{N} = \epsilon^{\gamma\beta} M^{\alpha}{}_{\beta}\mathbf{A}_\gamma + M^\alpha\mathbf{N}$ ([8], eqn (3.61), we set

$$a^{1/2}\mathbf{M}^s = M^{s}{}_{\theta}\mathbf{A}_s - M^{s}{}_{s}\mathbf{A}_\theta + a^{1/2}M^s\mathbf{N}$$

$$\equiv \bar{M}_{sr}\bar{\mathbf{e}}_r + \bar{M}_{s\theta}\bar{\mathbf{e}}_\theta + \bar{M}_s\mathbf{e}_z \tag{7.4}$$

$$a^{1/2}\mathbf{M}^\theta = M^{\theta}{}_{\theta}\mathbf{A}_s - M^{\theta}{}_{s}\mathbf{A}_\theta + a^{1/2}M^\theta\mathbf{N} \equiv \bar{M}_{\theta r}\bar{\mathbf{e}}_r + \bar{M}_{\theta\theta}\bar{\mathbf{e}}_\theta + \bar{M}_\theta\mathbf{e}_z. \tag{7.5}$$

Resolving $\bar{\mathbf{r}}' \times \mathbf{N}^s$ and $\bar{\mathbf{r}}^{\cdot} \times \mathbf{N}^\theta$ into components in the basis $(\bar{\mathbf{e}}_r, \bar{\mathbf{e}}_\theta, \mathbf{e}_z)$ and using (5.1), (5.2), (5.5), (6.10) and (6.11) and the differentiation formulas (3.49) and (3.50), we obtain from (7.2) the following equations in the $\bar{\mathbf{e}}_r$ and $\bar{\mathbf{e}}_\theta$ directions

$$\bar{M}'_{sr} + \gamma'\bar{M}_{s\theta} - (1-2q)\left[\bar{M}_{\theta\theta} + F_r\bar{z}' + \bar{a}_{s\theta}\left(\int_0^s a^{1/2}q_z\, dt - C\right)\right] + (F'_\theta - \gamma' F_r - a^{1/2}q_r)\bar{b} - \bar{a}_{\theta\theta}F'_z = 0 \tag{7.6}$$

$$\bar{M}'_{s\theta} - \gamma'\bar{M}_{sr} + (1-2q)\left[\bar{M}_{\theta r} - F_\theta\bar{z}' + \bar{a}_{sr}\left(\int_0^s a^{1/2}q_z\, dt - C\right)\right] + \bar{a}_{\theta r}F'_z - (F'_r + \gamma' F_\theta + a^{1/2}q_\theta)\bar{b} = 0. \tag{7.9}$$

In addition, the above component representations give the first integral (7.3) the more explicit form

$$M^{s}{}_{\theta}\sin\Phi - M^{s}{}_{s}(r\sin\psi\sin\alpha + b\cos\psi) + a^{1/2}M^s\cos\Xi$$
$$+ \bar{a}_{\theta\theta}F_r - \bar{a}_{\theta r}F_\theta + \int_0^s (\bar{a}_{\theta r}q_r + \bar{a}_{\theta\theta}q_\theta)a^{1/2}\, dt = (1-2q)B. \tag{7.10}$$

8. CONSTITUTIVE LAWS

To obtain a complete set of field equations, the compatibility conditions and moment equilibrium equations must be supplemented by constitutive laws. Obviously, if the stresses and strains are.to depend on s only, so must any nonhomogeneities.

Appropriate stress and conjugate strain measures may be inferred from expression for the virtual work per unit area of the undeformed midsurface. From eqn (3.65) of [8], this density, specialized to our problem, takes the form

$$N^{ss}\delta E_{ss} + 2N\delta E + N^{\theta\theta}\delta E_{\theta\theta} + Q^s\delta E_s + Q^\theta\delta E_\theta + \bar{N}\delta\bar{E}$$
$$+ M^{ss}\delta K_{ss} + 2M\delta K + M^{\theta\theta}\delta K_{\theta\theta} + M^s\delta K_s + M^\theta\delta K_\theta + \bar{M}\delta\bar{K}, \tag{8.1}$$

where

$$N = \tfrac{1}{2}(N^{s\theta} + N^{\theta s}),\ \bar{N} = \tfrac{1}{2}a^{1/2}(N^{s\theta} - N^{\theta s}) \tag{8.2}$$

$$E = \tfrac{1}{2}(E_{s\theta} + E_{\theta s}),\ \bar{E} = \tfrac{1}{2}a^{-1/2}(E_{s\theta} - E_{\theta s}). \tag{8.3}$$

with analogous expressions for M, $\bar{M}$, K, and $\bar{K}$. We have used co- and contravariant components in (8.1)–(8.3) as the corresponding forms with mixed components are a bit more complicated.

The strain energy per unit area of the undeformed midsurface has the form

$$\Phi = \hat{\Phi}(E_{ss}, E, \dots, \bar{K}) \tag{8.4}$$

so that

$$N^{ss} = \frac{\partial\Phi}{\partial E_{ss}},\ N = \frac{\partial\Phi}{\partial E}, \dots, \bar{M} = \frac{\partial\Phi}{\partial\bar{K}}. \tag{8.5}$$

As the components of $\boldsymbol{\beta}$ and $\mathbf{F}$ are the basic unknowns, we assume that (8.5) can be solved for the extensional, shear, and twist strains as functions of the bending strains and stress resultants. Then, by a Legendre transformation, we may introduce the *mixed* energy density

$$\Psi = \Phi - (N^{ss}E_{ss} + 2NE + \cdots + \bar{N}\bar{E}) \tag{8.6}$$

so that

$$E_{ss} = -\frac{\partial\Psi}{\partial N^{ss}},\ E = -\frac{\partial\Psi}{\partial N}, \dots, \bar{M} = \frac{\partial\Psi}{\partial\bar{K}}. \tag{8.7}$$

In the classical, small strain theory of elastically isotropic shells,

$$\begin{aligned}\Psi = {} & \tfrac{1}{2}D[K^{ss}K_{ss} + K^{\theta\theta}K_{\theta\theta} + 2\mu K^{ss}K_{\theta\theta} + 2(1-\mu)K^2] \\ & -\tfrac{1}{2}A[N^{ss}N_{ss} + N^{\theta\theta}N_{\theta\theta} - 2\nu N^{ss}N_{\theta\theta} + 2(1+\nu)N^2],\end{aligned} \tag{8.8}$$

where D is the bending stiffness, A is the stretching compliance, and μ and ν are Poisson ratios of bending and stretching, all depending, in general, on s. Conventionally,

$$D = \frac{Eh^3}{12(1-\nu^2)},\ A = \frac{1}{Eh},\ \mu = \nu, \tag{8.9}$$

where E is Young's modulus and h is the shell thickness. When (8.7) is substituted into the compatibility and moment equilibrium conditions, we obtain 6 equations for α, γ, ψ, F_r, F_θ and F_z.

9. SPECIAL CASES

The coordinates (s, θ) on the undeformed midsurface are orthogonal if b or φ is zero. If $b = 0$, we have an (incomplete) shell of revolution: if $\phi = 0$, a right helicoidal shell. Reissner [5–7] and Wan [14] have studied special nonlinear boundary value problems for these respective shells. We now show how our equations reduce to theirs.

Pure bending of a tube of arbitrary cross-section

To obtain Reissner's equations [7], set

$$b = \alpha = \gamma = 0. \tag{9.1}$$

Then $a^{1/2} = r$ and from (4.5) to (4.7) and (5.10) to (5.12) we get

$$\cos\Xi = \cos\Lambda = \cos\Phi,\ \cos Z = 0,\ \cos\Omega = -\sin\Phi, \tag{9.2}$$

where

$$\Phi = \phi + \psi. \tag{9.3}$$

Equations (4.9), (4.10), and (4.13) now reduce to

$$K_{s.}{}^{s} = \psi', \; rK_{\theta.}{}^{\theta} = (1-2q)\sin\Phi - \sin\phi, \; K_{\theta} = (1-2q)\cos\Phi - \cos\phi. \tag{9.4}$$

These bending strains agree with equations (14a, b) of [7]. If $q = 0$, we obtain the bending strains for the torsionless, axisymmetric deformation of a shell of revolution.

If we assume further that $E_{\theta.}{}^{s} = E_{s.}{}^{\theta} = E_{\theta} = 0$, then (5.13)–(5.16) reduce to

$$\bar{a}_{sr} = (1 + E_{s.}{}^{s})\cos\Phi - E_{s}\sin\Phi, \; \bar{a}_{\theta\theta} = r(1 + E_{\theta.}{}^{\theta}), \tag{9.5}$$

$$\bar{a}_{s\theta} = \bar{a}_{\theta r} = 0. \tag{9.6}$$

The compatibility conditions (5.6) and (5.17) are satisfied identically, while (5.7) reduces to

$$[r(1 + E_{\theta.}{}^{\theta})]' - (1-2q)[1 + E_{s.}{}^{s})\cos\Phi - E_{s}\sin\Phi] = 0. \tag{9.7}$$

This equation agrees with eqn (15) of [7], if it is noted that, because Reissner's normal to the deformed midsurface is the negative of ours, his shearing strain is the negative of ours.

To get Reissner's static equations, set

$$F_{r} = F_{z} = 0, \; \mathbf{p} = 0. \tag{9.8}$$

Then with the aid of (4.4) and (5.10) to (5.12), (6.10) and (6.11) reduce to

$$\begin{aligned} r\mathbf{N}^{s} &= -(1-2q)F_{\theta}\bar{\mathbf{e}}_{r} + C\mathbf{e}_{z} \\ &= [-(1-2q)F_{\theta}\cos\Phi + C\sin\Phi]\mathbf{A}_{s} + [(1-2q)F_{\theta} + C\cos\Phi]\mathbf{N} \end{aligned} \tag{9.9}$$

$$r\mathbf{N}^{\theta} = -F'_{\theta}\bar{\mathbf{e}}_{\theta} = -F'_{\theta}\mathbf{A}_{\theta}. \tag{9.10}$$

These expressions agree with eqns (28) of [7].

The above assumptions plus $M^{s}{}_{\theta}^{\cdot} = M^{\theta}{}_{s}^{\cdot} = M^{s} = 0$ imply, by (4.4), (5.10)–(5.12), (7.4) and (7.5) that

$$\bar{M}_{s\theta} = -rM^{s}{}_{s}^{\cdot}, \; \bar{M}_{\theta r} = M^{\theta}{}_{\theta}^{\cdot}\cos\Phi - M^{\theta}\sin\Phi \tag{9.11}$$

$$\bar{M}_{sr} = \bar{M}_{\theta\theta} = 0. \tag{9.12}$$

The moment equilibrium equations (7.6) and (7.10) are satisfied identically, while (7.9) reduces to

$$\begin{aligned} -(rM^{s}{}_{s}^{\cdot})' + (1-2q)[M^{\theta}{}_{\theta}^{\cdot}\cos\Phi - M^{\theta}\sin\Phi \\ -(1 + E_{s.}{}^{s})(C\cos\Phi + F_{\theta}\sin\Phi) + E_{s}(C\sin\Phi - F_{\theta}\cos\Phi)] = 0. \end{aligned} \tag{9.13}$$

If the normal stress couple component M^{θ} is set to zero. (9.13) reduces to eqns (21) and (28) of [7], if it is noted that Reissner's shear strain γ and transverse shear stress resultant Q, by virtue of his definition of the normal to the deformed midsurface, have the opposite sign as ours.

Extension and twist of a right helicoidal shell

We follow Wan [14] and use a semi-inverse method to obtain the kinematic and static equations governing the gross extension and twist of a right helicoidal shell ($r = c + s$, $\phi = 0$), stress free along its radial edges $s = 0$ and $s = L$. That is, we retain only enough freedom among the dependent variables to allow for the imposition of a net axial force P and a net axial torque T over any edge θ = constant. We assume that the constitutive laws are consistent with our *a priori* assumptions, but do not attempt to give them explicit form. For a linear analysis using the Kirchhoff Hypothesis, see Reissner and Wan [15].

We first assume that

$$\alpha = \pi/2,\ \gamma = 0. \tag{9.14}$$

With

$$r = a^{1/2}\cos\chi,\ b = a^{1/2}\sin\chi, \tag{9.15}$$

(4.5)–(4.7) and (5.10)–(5.12) reduce to

$$\sin\Phi = \cos Z = \cos\Lambda = 0,\ \cos\Xi = \cos X,\ \cos\Omega = -\sin X, \tag{9.16}$$

where

$$X = \chi + \psi. \tag{9.17}$$

The bending strains follow from (4.8) to (4.13) as

$$K_{s.}{}^{\theta} = a^{-1/2}\psi',\ K_{\theta.}{}^{s} = \sin\chi - (1-2q)\sin X,\ K_{\theta} = (1-2q)\cos X - \cos\chi \tag{9.18}$$

$$K_{s.}{}^{s} = K_{\theta.}{}^{\theta} = K_{s} = 0. \tag{9.19}$$

With the further assumption that

$$E_{s.}{}^{\theta} = E_{\theta.}{}^{s} = E_{s} = 0, \tag{9.20}$$

(5.13)–(5.16) yield

$$\bar{a}_{s\theta} = -(1 + E_{s.}{}^{s}),\ \bar{a}_{\theta r} = a^{1/2}[(1 + E_{\theta.}{}^{\theta})\cos X - \gamma_{\theta}\sin X] \tag{9.21}$$

$$\bar{a}_{sr} = \bar{a}_{\theta\theta} = 0, \tag{9.22}$$

where $\gamma_{\theta} = a^{-1/2}E_{\theta}$ is the physical component of the non-zero transverse shearing strain. The compatibility condition (5.7) is satisfied identically while (5.6) and (5.17) reduce to

$$\{a^{1/2}([1 + E_{\theta.}{}^{\theta})\cos X - \gamma_{\theta}\sin X]\}' - (1-2q)(1 + E_{s.}{}^{s}) = 0 \tag{9.23}$$

$$a^{1/2}[(1 + E_{\theta.}{}^{\theta})\sin X + \gamma_{\theta}\cos X] = \bar{b}. \tag{9.24}$$

To obtain static equations, we assume that that

$$\mathbf{p} = 0,\ F_{\theta} = C = M^{s}{}_{;s} = M^{\theta}{}_{;\theta} = M_{s\theta} - M_{\theta s} = M^{s} = 0, \tag{9.25}$$

the last four conditions being constitutive assumptions. Then, with

$$M \equiv \tfrac{1}{2}(M_{s\theta} + M_{\theta s}), \tag{9.26}$$

(4.4), (5.10)–(5.12), (7.4) and (7.5) imply that

$$\bar{M}^{s}_{\theta} = M^{s}{}_{;\theta}\mathbf{A}_{s}\cdot\bar{\mathbf{e}}_{\theta} = -a^{1/2}M,\ \bar{M}_{\theta r} = -M^{\theta}{}_{;s}\mathbf{A}_{\theta}\cdot\bar{\mathbf{e}}_{r} = -M\cos X. \tag{9.27}$$

The moment equilibrium equations (7.6) and (7.10) are satisfied identically while (7.9) reduces to

$$(a^{1/2}M)' + (1-2q)M\cos X + a^{1/2}[(1 + E_{\theta.}{}^{\theta})(F'_{r}\sin X - F'_{z}\cos X) + F'_{r}\gamma_{\theta}\cos X] = 0, \tag{9.28}$$

Overall force equilibrium along any edge θ = constant requires that

$$\int_0^L a^{1/2}\mathbf{N}^\theta \, ds = P\mathbf{e}_z, \tag{9.29}$$

which, by (6.11) and (9.25), implies

$$F_r(L) - F_r(0) = 0, \; F_z(L) - F_z(0) = -P. \tag{9.30}$$

Overall moment equilibrium along any edge θ = constant requires that

$$\int_0^L a^{1/2}(\mathbf{M}^\theta + \bar{\mathbf{r}} \times \mathbf{N}^\theta) \, ds = T\mathbf{e}_z, \tag{9.31}$$

which, with the aid of (5.20), (7.5), (9.21) and (9.22), leads to the two scalar conditions

$$\int_0^L \{(1-2q)(M\cos X + m_\theta \sin X) - a^{1/2}[(1+E_{\theta\cdot}{}^\theta)\cos X - \gamma_\theta \sin X]F_z'\} \, ds = 0 \tag{9.32}$$

$$\int_0^L \{(1-2q)(M\sin X - m_\theta \cos X) + a^{1/2}[(1+E_{\theta\cdot}{}^\theta)\cos X - \gamma_\theta \sin X]F_r'\} \, ds = -(1-2q)T, \tag{9.33}$$

a third condition being identical to $(9.30)_1$. In (9.32) and (9.33) $m_\theta = a^{1/2}M^\theta$ is the physical component of the non-zero normal stress couple.

Equations (6.10) and (7.4) plus the assumptions made in this section imply that the edges $s = 0, L$ will be stress free if

$$F_r(L) = F_r(0) = M(L) = M(0) = 0. \tag{9.34}$$

A complete set of field equations follows upon specifying the mixed energy density

$$\Psi = \hat{\Psi}(N^s{}_{;s}, N^\theta{}_{;\theta}, Q^\theta, K, K_\theta) = \bar{\Psi}(F_r, F_z, \psi, q, \bar{b}), \tag{9.35}$$

where,

$$K \equiv \tfrac{1}{2}(K_{s\theta} + K_{\theta s}) = \tfrac{1}{2}[a^{1/2}\psi' + \sin\chi - (1-2q)\sin X]. \tag{9.36}$$

10. INEXTENSIONAL DEFORMATION

We extend Reissner's results for the inextensional deformation of split shells of revolution [16] to general helicoids. Interestingly, it is of no help to determine the rotation field first. Rather, we obtain $\bar{r}$, τ, and $\bar{z}$ directly, starting from (1.3).

With

$$\hat{\mathbf{e}}_r = \cos(\lambda\theta - \tau)\mathbf{e}_x + \sin(\lambda\theta - \tau)\mathbf{e}_y \tag{10.1}$$

$$\hat{\mathbf{e}}_\theta = -\sin(\lambda\theta - \tau)\mathbf{e}_x + \cos(\lambda\theta - \tau)\mathbf{e}_y, \tag{10.2}$$

we have

$$\mathbf{A}_s = \bar{\mathbf{a}}_s = \mathbf{r}' = \bar{r}'\hat{\mathbf{e}}_r - \bar{r}\tau'\hat{\mathbf{e}}_\theta + \sin\Phi\mathbf{e}_z \tag{10.3}$$

$$\mathbf{A}_\theta = \bar{\mathbf{a}}_\theta = \mathbf{r}^{\cdot} = \lambda\bar{r}\hat{\mathbf{e}}_\theta + \bar{b}\mathbf{e}_z, \tag{10.4}$$

where, from (4.5),

$$\bar{z}' = \mathbf{A}_s \cdot \mathbf{e}_z = \sin\Phi. \tag{10.5}$$

The deformation being inextensional, we have, by (2.1) and (2.2),

$$a_{ss} \equiv \mathbf{a}_s \cdot \mathbf{a}_s = 1 = \bar{\mathbf{a}}_s \cdot \bar{\mathbf{a}}_s = \bar{r}'^2 + (\bar{r}\tau')^2 + \sin^2\Phi \tag{10.6}$$

$$a_{s\theta} \equiv \mathbf{a}_s \cdot \mathbf{a}_\theta = b \sin\phi = \bar{\mathbf{a}}_s \cdot \bar{\mathbf{a}}_\theta = -\lambda\bar{r}^2\tau' + \bar{b}\sin\Phi \tag{10.7}$$

$$a_{\theta\theta} \equiv \mathbf{a}_\theta \cdot \mathbf{a}_\theta = r^2 + b^2 = \bar{\mathbf{a}}_\theta \cdot \bar{\mathbf{a}}_\theta = (\lambda\bar{r})^2 + \bar{b}^2. \tag{10.8}$$

These are 3 nonlinear equations for $\bar{r}$, Φ, and τ.

From (10.8),

$$\lambda\bar{r} = (r^2 + b^2 - \bar{b}^2)^{1/2}. \tag{10.9}$$

Thence, from (10.7)

$$\bar{r}\tau' = \frac{\bar{b}\sin\Phi - b\sin\phi}{(r^2 + b^2 - \bar{b}^2)^{1/2}}. \tag{10.10}$$

To get $\sin\Phi$, we first differentiate (10.9):

$$\bar{r}' = \frac{r\cos\phi}{\lambda(r^2 + b^2 - \bar{b}^2)^{1/2}}. \tag{10.11}$$

Then, substituting (10.10) and (10.11) into (10.6), we get a quadratic equation for $\sin\Phi$ whose solution is

$$\sin\Phi = \frac{b\bar{b}\sin\rho}{r^2 + b^2} \pm \left[\frac{a(r^2 + b^2 - \bar{b}^2)}{(r^2 + b^2)^2} - \frac{r^2\cos^2\phi}{\lambda^2(r^2 + b^2)}\right]^{1/2}, \tag{10.12}$$

where a is given by (2.6).

Equations (10.9) and (10.12) are explicit formulas for $\bar{r}$ and $\sin\Phi$. Substituted into (10.5) and (10.10), they reduce the determination of $\bar{z}$ and τ to quadratures.

The bending strains may be computed as the differences in the curvature tensors of the deformed and undeformed midsurface. However, it is simpler to note, first, that (4.10), (4.11), and (5.17) give

$$a^{1/2}K_{\theta.}^{\theta} = \lambda\sin\Phi - \sin\phi \tag{10.13}$$

$$a^{1/2}K_{\theta.}^{s} = b - \lambda\bar{b}. \tag{10.14}$$

Then,

$$\begin{aligned} K_{s\theta} = K_{\theta s} &= K_{\theta.}^{s}a_{ss} + K_{\theta.}^{\theta}a_{s\theta} \\ &= a^{-1/2}[b(\cos^2\rho + \lambda\sin\Phi\sin\rho) - \lambda\bar{b}] \end{aligned} \tag{10.15}$$

$$\begin{aligned} K_{\theta\theta} &= K_{\theta.}^{s}a_{s\theta} + K_{\theta.}^{\theta}a_{\theta\theta} \\ &= a^{-1/2}[(b - \lambda\bar{b})b\sin\phi + (\lambda\sin\Phi - \sin\phi)(r^2 + b^2)]. \end{aligned} \tag{10.16}$$

To compute K_{ss} (and thence $K_{s.}^{s}$), we follow Reissner [16] and note that, as the Gaussian curvature G is unchanged in an inextensional deformation,

$$aG = b_{ss}b_{\theta\theta} - b_{s\theta}^2 = (b_{ss} + K_{ss})(b_{\theta\theta} + K_{\theta\theta}) - (b_{s\theta} + K_{s\theta})^2, \tag{10.17}$$

i.e.

$$K_{ss} = \frac{(2b_{s\theta} + K_{s\theta})K_{s\theta} - b_{ss}K_{\theta\theta}}{b_{\theta\theta} + K_{\theta\theta}}. \tag{10.18}$$

Here, from (2.1) to (2.3),

$$b_{ss} = \mathbf{n}\cdot\mathbf{a}_s = a^{-1/2} r\phi' \tag{10.19}$$

$$b_{s\theta} = \mathbf{n}\cdot\mathbf{a}_s^{\cdot} = \mathbf{n}\cdot\mathbf{a}_\theta' = -a^{-1/2} b\cos^2\phi \tag{10.20}$$

$$b_{\theta\theta} = \mathbf{n}\cdot\mathbf{a}_\theta^{\cdot} = a^{-1/2} r^2 \sin\phi \tag{10.21}$$

Finally,

$$K_{s.}^{\ w} = K_{ss}a^{ss} + K_{s\theta}a^{s\theta},\quad K_{s.}^{\ \theta} = K_{ss}a^{s\theta} + K_{s\theta}a^{\theta\theta}, \tag{10.22}$$

where

$$a^{ss} = a^{-1}(r^2+b^2),\ a^{s\theta} = -a^{-1}b\sin\phi,\ a^{\theta\theta} = a^{-1}. \tag{10.23}$$

Substituting (10.15), (10.16) and (10.19)–(10.21) into (10.18) and the resulting equation along with (10.15) into (10.22), we obtain

$$K_{s.}^{\ s} = a^{-3/2}\left\{\frac{a[a(r^2+b^2)G + \lambda^2\bar{b}(\bar{b} - b\sin\Phi\sin\phi)]}{\lambda[(r^2+b^2)\sin\Phi - b\bar{b}\sin\phi]} - (r^2+b^2)r\phi' - b^2\sin\phi\cos^2\phi\right\} \tag{10.24}$$

$$K_{\theta.}^{\ s} = -a^{-3/2}\left\{\frac{a[abG\sin\phi - \lambda^2\sin\Phi(b\sin\Phi\sin\phi - \bar{b})]}{\lambda[(r^2-b^2)\sin\Phi - b\bar{b}\sin\phi]} - b(r\phi'\sin\phi + \cos^2\phi)\right\}, \tag{10.25}$$

where

$$a^2G = r^3\phi'\sin\phi - b^2\cos^4\phi. \tag{10.26}$$

If we set $b = 0$, our equations reduce to those of Reissner [16] for a slit shell of revolution. If we set $\phi = \pi/2$ (in which case r is a constant), we obtain equations for a cylindrical helicoidal shell, which has been studied by Mansfield[17]. If we set $\phi = 0$ and $r = c + s$, we obtain the following displacement and bending fields for the inextensional deformation of a right helicoidal shell.

$$\bar{r} = \lambda^{-1}(r^2 + b^2 - \bar{b}^2)^{1/2} \tag{10.27}$$

$$\sin\Phi = \pm\lambda^{-1}a^{-1/2}[\lambda^2(r^2+b^2-\bar{b}^2) - r^2]^{1/2} \tag{10.28}$$

$$\tau = \lambda\bar{b}\int (r^2+b^2-\bar{b}^2)^{-1}\sin\Phi\, dr \tag{10.29}$$

$$\bar{z} = \int \sin\Phi\, dr \tag{10.30}$$

$$K_{\theta.}^{\ \theta} = \frac{\lambda\sin\Phi}{(r^2+b^2)^{1/2}},\ \mathbf{K}_{\theta.}^{\ s} = \frac{b - \lambda\bar{b}}{(r^2+b^2)^{1/2}} \tag{10.31}$$

$$K_{s.}^{\ s} = \frac{\lambda^2\bar{b}^2 - b^2}{\lambda(r^2+b^2)^{3/2}},\ K_{\theta.}^{\ s} = \frac{\lambda\bar{b} - b\sin\Phi}{(r^2+b^2)^{3/2}}. \tag{10.32}$$

REFERENCES

1. J. Simmonds, Surfaces with metric and curvature tensors that depend on one coordinate only are general helicoids. *Q. Appl. Math.* **37**, 82–85 (1979).
2. E. Reissner, On the equations for the finite symmetrical deflections of thin shells of revolution. In *Progress in Solid Mechanics* (The Prager Anniversary Volume), pp. 171–178. Macmillan, New York (1963).
3. E. Reissner, On the symmetrical deflections of thin shells of revolution. *J. Appl. Mech.* **36**, 267–270 (1969).

4. E. Reissner, On finite symmetrical strain in thin shells of revolution. *J. Appl. Mech.* **39**, 1137 (1972).
5. E. Reissner, On the finite bending of pressurized tubes. *J. Appl. Mech.* **26**, 69–72 (1959).
6. E. Reissner, On finite pure bending of cylindrical tubes. *Österreichisches Ingenieur* **15**, 165–172 (1961).
7. E. Reissner, On finite pure bending of curves tubes. *Int. J. Solids Structures* **17**, 839–844 (1981).
8. A. Libal and J. Simmonds, Nonlinear elastic shell theory. In *Advances in Applied Mechanics*, Vol 23 (Edited by Hutchinson and Wu), pp. 271–371. Academic Press, New York (1983).
9. M. Beatty, Vector analysis of finite rigid rotations. *J. Appl. Mech.* **44**, 501–502 (1977).
10. J. Simmonds and D. Danielson, Nonlinear shell theory with finite rotation and stress-function vectors. *J. Appl. Mech.* **39**, 1085–1090 (1972).
11. W. Pietraszkiewicz, *Finite Rotations and Lagrangian Description in the Non-Linear Theory of Shells*. Polish Scientific, Warsaw (1979).
12. E. Axelrad, On vector description of arbitrary deformation of shells, *Int. J. Solids Structures* **17**, 301–304 (1981).
13. J. Simmonds, *A Brief on Tensor Analysis*, Springer-Verlag, New York (1982).
14. F. Wan, Nonlinear torsion and extension of helicoidal shells Submitted for publication.
15. E. Reissner and F. Wan, On axial extension and torsion of helicoidal shells. *J. Math. Phys.* **47**, 1–31 (1968).
16. E. Reissner, Finite inextensional pure bending and twisting of thin shells of revolution. *Q. J. Mech. Appl. Math.* **21**, 293–306 (1968).
17. E. Mansfield, On finite inextensional deformation of a helical strip. *Int. J. Non-linear Mech.* **15**, 459–467 (1980).